第7章　積分法とその応用

不定積分の公式

$$\int x^{\alpha}\,dx = \frac{1}{\alpha+1}x^{\alpha+1} + C$$

$$\int \frac{1}{x}\,dx = \log|x| + C$$

$$\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$$

$$\int \sqrt{x}\,dx = \frac{2}{3}x\sqrt{x} + C$$

$$\int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x} + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \frac{1}{\cos^2 x}\,dx = \tan x + C$$

$$\int \frac{1}{\sin^2 x}\,dx = -\frac{1}{\tan x} + C$$

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\log a} + C$$

$$\int \log x\,dx = x\log x - x + C$$

$\boxed{1}$ ～ $\boxed{3}$

置換積分法（基本）

$$\boxed{\int f(x)\,dx = \int f(g(t))\,g'(t)\,dt}$$

ex

$$\int \underline{x}\ \underline{(2x-1)^3}\ \underline{dx}$$

$\boxed{2x-1} = \boxed{t}$ とおく

$$\boxed{x} = \boxed{\frac{t+1}{2}}$$

$$\frac{dx}{dt} = \frac{1}{2}$$

$$\boxed{dx} = \boxed{\frac{1}{2}\,dt}$$

$$\int \underline{\frac{t+1}{2}}\ \underline{t^3}\ \underline{\frac{1}{2}\,dt} = \frac{1}{4}\int (t^4+t^3)\,dt$$

$$= \frac{1}{4}\left(\frac{1}{5}t^5 + \frac{1}{4}t^4\right) + C$$

$$= \frac{1}{80}\underline{t^4}(4\underline{t}+5) + C$$

$$= \frac{1}{80}\underline{(2x-1)^4}(8x+1) + C$$

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置換する文字

$\boxed{5}$ (3) (別解)

$$\int \frac{x}{\sqrt{x+1}}\,dx$$

※ $\sqrt{x+1} = t$ が一般的

$x+1 = t$ とおく　$x = t-1$

$dx = dt$

$$\int \frac{t-1}{\sqrt{t}}\,dt = \int \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) dt$$

$$= \int \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) dt$$

$$= \frac{2}{3} t^{\frac{3}{2}} - 2t^{\frac{1}{2}} + C$$

$$= \frac{2}{3} t^{\frac{1}{2}} (t-3) + C$$

$$= \frac{2}{3} \sqrt{x+1}\,(x-2) + C \quad //$$

同じ答えになる

$(●)^{\alpha}$ の形は $● = t$ とおくことが多い。

$\sqrt[n]{●}$ の形は $\sqrt[n]{●} = t$ とおくことが多い

$● \times \sqrt[n]{●}$ の形は $● = t$ とおくことが多い

置換積分法② (1次式)

$$\boxed{\int f(ax+b)\,dx = \frac{1}{a}F(ax+b)+C}$$

$f(x)$ の x に 1次式 $ax+b$ を代入した $f(ax+b)$ の積分

x の係数の逆数 $\frac{1}{a}$ をかけて、いつも通り積分する

ex

(1) $\displaystyle\int (4x+1)^2\,dx = \frac{1}{4}\cdot\frac{1}{3}(4x+1)^3+C$

$\displaystyle = \frac{1}{12}(4x+1)^3+C$ //

(2) $\displaystyle\int \frac{dx}{\cos^2(5x-2)} = \frac{1}{5}\tan(5x-2)+C$ //

$$\boxed{\int \frac{1}{\cos^2 x} = \tan x + C}$$

(3) $\displaystyle\int 5^{-\frac{3}{2}x+2}\,dx = -\frac{2}{3}\,\frac{5^{-\frac{3}{2}x+2}}{\log 5}$

$$\boxed{\int a^x dx = \frac{a^x}{\log a}+C}$$

$\displaystyle = -\frac{2\cdot 5^{-\frac{3}{2}x+2}}{3\log 5}+C$ //

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置換積分法 ③　$g\,g'$

$$\boxed{\int f(g(x))\,g'(x)\,dx = \int f(t)\,dt}$$

一部を $g(x)$ とすると，他の一部が $g'(x)$ で表されるとき

⇩

$g(x)$ を t とおく

ex

(1) $\displaystyle\int x^2\sqrt{x^3+2}\,dx = \frac{1}{3}\int \frac{1}{3}x^2\sqrt{\underbrace{\boxed{x^3+2}}_{=t}}$

(2) $\displaystyle\int \underbrace{\boxed{\sin^2 x}}_{=t}\cos x\,dx$

(3) $\displaystyle\int \frac{\overbrace{\boxed{\log x}}^{=t}}{x}\,dx$

(4) $\displaystyle\frac{\tan x}{(\underbrace{\boxed{\tan x-1}}_{=t})^2\cos^2 x}\,dx$

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置換積分法③-1 $\frac{g'}{g}$

$$\boxed{\int \frac{g'}{g}\,dx = \log|g| + C}$$

分母を微分すると分子になるとき

ex

(1) $\displaystyle \int \frac{6x}{3x^2-5}\,dx = \int \frac{(3x^2-5)'}{3x^2-5}\,dx$

$$= \log|3x^2-5| + C$$

(2) $\displaystyle \int \frac{1}{x(\log x+1)}\,dx = \frac{\frac{1}{x}}{\log x+1}\,dx$

$$= \log|\log x+1| + C$$

(3) $\displaystyle \int \frac{1}{\cos^2 x(\tan x+1)}\,dx = \int \frac{\frac{1}{\cos^2 x}}{\tan x+1}\,dx$ $\boxed{7}$

$$= \log|\tan x+1| + C$$

置換積分法 ③-2　　$g^{\alpha} g'$ (累乗)

$$\boxed{\int \{g(x)\}^{\alpha} g'(x)\,dx = \frac{\{g(x)\}^{\alpha+1}}{\alpha+1} + C}$$

$\int x^{\alpha}\,dx = \dfrac{x^{\alpha+1}}{\alpha+1} + C$ と同じように計算

ex

(1) $\int x(x^2+1)^5\,dx = \frac{1}{2}\boxed{\int 2x(x^2+1)^5\,dx}$

$= \frac{1}{2}\cdot\boxed{\frac{1}{6}(x^2+1)^6 + C}$

$= \underline{\frac{1}{12}(x^2+1)^6 + C}$ //

b (別解)

(2) $\int x^2\sqrt{x^3+2}\,dx = \frac{1}{3}\int 3x^2\sqrt{x^3+2}\,dx$

$= \frac{1}{3}\cdot\frac{2}{3}(x^3+2)\sqrt{x^3+2} + C$

$= \underline{\frac{4}{9}(x^3+2)\sqrt{x^3+2} + C}$ //

(3) $\int \sin^3 x\cos x\,dx = \underline{\frac{1}{4}\sin^4 x + C}$ //

b

部分積分法

$$\boxed{\int f g' \, dx = fg - \int f' g \, dx}$$

ex $\displaystyle \int x e^x dx = \int x (e^x)' dx = x e^x - \int 1 \cdot e^x dx$

$$= \underline{x e^x - e^x + C}$$

なぜなら

$$(fg)' = f'g + fg'$$

両辺積分

$$\int (fg)' dx = \int f'g \, dx + \int fg' \, dx$$

$$fg = \int f'g \, dx + \int fg' \, dx$$

$$\boxed{\int fg' \, dx = fg - \int f'g \, dx}$$

① 積分する関数が積の形

② f が微分すると簡単になる.

部分積分法のパターン

① 多項式 × 多項式　$x^2(5x+1)$

　　　　　× 指数関数　x^2e^x

　　　　　× 三角関数　$x^2\sin 2x$

② 多項式 × 対数関数　$x^3(\log x)^2$

③ 指数関数 × 三角関数　$e^{2x}\sin 5x$

fにする方

対数関数 > 多項式 > 三角関数 > 指数関数

$\log x > x^n > \sin x > e^x$

部分積分法のパターン①

多項式 × 多項式 / 指数関数 / 三角関数

ex

(1)
$$\int x \sin x \, dx$$

g を積分する

$$= \int x (-\cos x)' \, dx$$

$$= -x\cos x - \int 1 \cdot (-\cos x) \, dx$$

f を微分する

$$= -x\cos x + \sin x + C$$

(2)
$$\int x e^{-x} \, dx$$

g を積分する

$$= \int x (-e^{-x})' \, dx$$

$$= -xe^{-x} - \int 1 \cdot (-e^{-x})$$

f を微分する

$$= -xe^{-x} - e^{-x} + C$$

$$= -(x+1)e^{-x} + C$$

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部分積分法のパターン②

多項式 × 対数関数

ex

(1) $\int \log 2x \, dx$

$= \int \underset{g}{1} \cdot \underset{f}{\log 2x} \, dx$　　gをつくる

$= \int (x)' \cdot \log 2x \, dx$　　gを積分する

$= x \log 2x - \int x \cdot \frac{1}{2x} \cdot 2 \, dx$　　fを微分する

$= x \log 2x - x + C$ //

部分積分法のパターン③　指数関数 × 三角関数

ex

$$\boxed{\int (e^x)' \sin x \, dx}$$

g を積分する

$= e^x \sin x - \int e^x \cos x \, dx$　　f を微分する

$= e^x \sin x - \int (e^x)' \cos x \, dx$　　g を積分する

$= e^x \sin x - \left\{ e^x \cos x - \int e^x (-\sin x) \, dx \right\}$　　f を微分する

$= e^x \sin x - e^x \cos x - \boxed{\int e^x \sin x \, dx}$

移項する

$$2\boxed{\int e^x \sin x \, dx} = e^x (\sin x - \cos x)$$

$$\int e^x \sin x \, dx = \underline{\frac{1}{2} e^x (\sin x - \cos x) + C}$$

部分積分法を2回適用する

瞬間部分積分

そのまま　微分　微分　微分

$$\int f g\,dx = fg - fg + fg - fg + \cdots$$

積分　積分　積分　積分

ex

$$\int x^2 \sin x\,dx = \int x^2(-\cos x)\,dx$$

$$= -x^2\cos x + 2\int x\cos x\,dx$$

$$= -x^2\cos x + 2\int x(\sin x)'\,dx$$

$$= -x^2\cos x + 2\left\{x\sin x - \int \sin x\,dx\right\}$$

$$= \underline{-x^2\cos x + 2x\sin x + 2\cos x + C} \;/\!/$$

[別解]

そのまま　微分　微分

$$\int x^2 \sin x\,dx = x^2(-\cos x) - 2x(-\sin x) + 2\cos x + C$$

積分　積分　積分

$$= \underline{x^2\cos x + 2x\sin x + 2\cos x + C} \;/\!/$$

分数関数の不定積分　復習

(1) $\displaystyle\int \frac{(2x+1)^2}{x}\,dx = \int \frac{4x^2+2x+1}{x}\,dx$　分離

$\displaystyle = \int \left(4x + 2 + \frac{1}{x}\right) dx$

$= 2x^2 + 2x + \log|x| + C$

(2) $\displaystyle\int \frac{e^{2x}-1}{e^x-1}\,dx = \int \frac{(e^x-1)(e^x+1)}{e^x-1}\,dx$　約分できるなら因数分解

$\displaystyle = \int (e^x+1)\,dx$

$= e^x + x + C$

(3) $\displaystyle\int \frac{6x}{3x^2-5}\,dx = \log|3x^2-5| + C$

$\displaystyle\int \frac{g'}{g}\,dx = \log|g| + C$

(4) $\displaystyle\int \frac{x}{(2x-1)^4}\,dx$　$2x-1=t$ とおく

置換積分

分数関数の不定積分

① 分子の次数 ≧ 分母の次数 のとき

分子の次数を低くする。

② 分母が因数分解できるとき

部分分数分解

ex

(1) $\displaystyle\int \frac{x^2-1}{x+2}\,dx$

次数を低くする

$x-2$

$$\begin{array}{r|rrr} 1 \quad 2 & 1 & 0 & -1 \\ & 1 & 2 & \\ \hline & & -2 & -1 \\ & & -2 & -4 \\ \hline & & & 3 \end{array}$$

$$= \int \left(x-2 + \frac{3}{x+2} \right) dx$$

$$= \frac{1}{2}x^2 - 2x + 3\log|x+2| + C$$

(2) $\displaystyle\int \frac{3}{x^2+x-2}\,dx$

$$= \int \frac{3}{(x+2)(x-1)}\,dx$$

部分分数分解

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$$= \int \left(\frac{1}{x-1} - \frac{1}{x+2} \right) dx$$

$$= \log|x-1| - \log|x+2| + C = \log\left|\frac{x-1}{x+2}\right| + C$$

部分分数分解①

(1) $$\frac{x+1}{x^2-3x+2} = \frac{x+1}{(x-2)(x-1)} = \frac{\boxed{3}}{x-2} \boxed{-\frac{2}{x-1}}$$

$$\left(\boxed{\frac{x+1}{x^2-3x+2} = \frac{A}{x-2} + \frac{B}{x-1}} \right.$$

$$x+1 = (x-1)A + B(x-2)$$

$x=1$のとき　$B = \boxed{-2}$

$x=2$のとき　$A = \boxed{3}$)

(2) $$\frac{1}{x(x+1)(x+2)} = \boxed{\frac{1}{2}}\frac{1}{x} \boxed{-}\frac{1}{x+1} + \boxed{\frac{1}{2}}\frac{1}{(x+2)}$$

$$\left(\boxed{\frac{1}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}} \right.$$

$A = \boxed{\frac{1}{2}}$　$B = \boxed{-1}$　$C = \boxed{\frac{1}{2}}$)

部分分数分解②

(3) $$\frac{x+2}{(x-1)(x^2+x+1)} = \frac{1}{x-1} - \frac{x+1}{x^2+x+1}$$

$$\left(\boxed{\frac{x+2}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}} \right.$$

$$x+2 = A(x^2+x+1) + (Bx+C)(x-1)$$

$x=1$ を代入　$A=1$

$x=0$ を代入　$C=-1$

$\left. x=2 \text{ を代入}\quad B=-1 \right)$

(4) $$\frac{1}{x^2(1-x)} = \frac{1}{x^2} + \frac{1}{x} + \frac{1}{1-x}$$

$$\left(\boxed{\frac{1}{x^2(1-x)} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{1-x}} \right.$$

$\left. A=1 \quad B=1 \quad C=1 \right)$

三角関数の不定積分

三角形の相互関数

① $\sin^2 x + \cos^2 x = 1$

② $\mathrm{Tan}\,\alpha = \dfrac{\sin x}{\cos x}$

③ $1 + \tan^2 x = \dfrac{1}{\cos^2 x}$

④ $1 + \dfrac{1}{\tan^2 x} = \dfrac{1}{\sin^2 x}$

倍角

① $\sin x \cos x = \dfrac{1}{2}\sin 2x$

② $\sin^2 x = \dfrac{1}{2}(1 - \cos 2x)$

③ $\cos^2 x = \dfrac{1}{2}(1 + \cos 2x)$

3倍角

① $\sin^3 x = \dfrac{1}{4}(3\sin x - \sin 3x)$

② $\cos^3 x = \dfrac{1}{4}(\cos 3x + 3\cos x)$

積→和

① $\sin\alpha\cos\beta = \dfrac{1}{2}\{\sin(\alpha+\beta) + \sin(\alpha-\beta)\}$

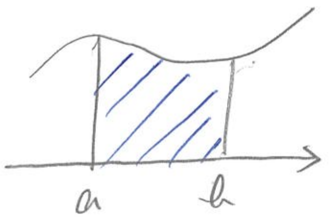
② $\cos\alpha\sin\beta = \dfrac{1}{2}\{\sin(\alpha+\beta) - \sin(\alpha-\beta)\}$

③ $\cos\alpha\cos\beta = \dfrac{1}{2}\{\cos(\alpha+\beta) + \cos(\alpha-\beta)\}$

④ $\sin\alpha\sin\beta = -\dfrac{1}{2}\{\cos(\alpha+\beta) - \cos(\alpha-\beta)\}$ $\boxed{11}$

定積分

$$\int_a^b f(x)\,dx$$



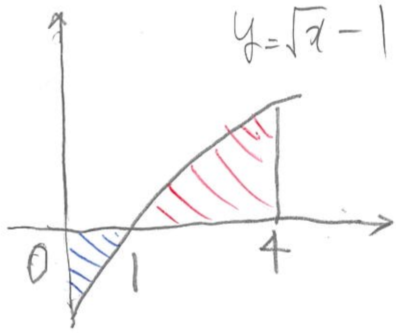
$\boxed{12}$ $\boxed{13}$

絶対値のついた定積分

ex

$$\int_0^4 |\sqrt{x}-1|\,dx$$

$$= -\int_0^1 (\sqrt{x}-1)\,dx + \int_1^4 (\sqrt{x}-1)\,dx$$



$$= -\left[\frac{2}{3}x\sqrt{x}-x\right]_0^1 + \left[\frac{2}{3}x\sqrt{x}-x\right]_1^4$$

$$= -\left\{\left(\frac{2}{3}-1\right)-0\right\} + \left(\frac{16}{3}-4\right)-\left(\frac{2}{3}-1\right)$$

$$= 2$$ //

別解

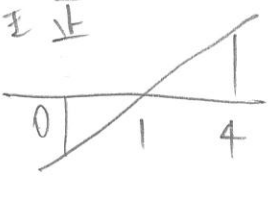
$$\int_0^4 |\sqrt{x}-1|\,dx$$

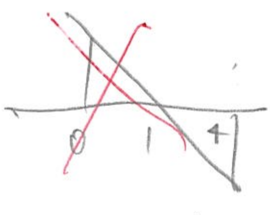
$\sqrt{x}-1=0$

$x=1$

$x=\frac{1}{2}$ のとき負

$x=2$ のとき正





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置換積分

ex

$$\int_0^1 \boxed{x}\,\boxed{(1-x)}^5\,dx$$

$\boxed{1-x} = \boxed{t}$ とおく

$$\boxed{x} = \boxed{1-t}$$

$$\frac{dx}{dt} = -1$$

$$dx = -dt$$

x	$0 \longrightarrow 1$
t	$1 \longrightarrow 0$

積分区間も変わる.

$$\int_1^0 \boxed{(1-t)}\,\boxed{t}^5\,\boxed{-}\,dt$$

$$\boxed{\int_a^b dx = -\int_b^a dx}$$

$$= \int_0^1 (1-t)\,t^5\,dt$$

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$\sqrt{a^2-x^2}$, $\dfrac{1}{\sqrt{a^2-x^2}}$ の定積分 ①

$\boxed{\sqrt{a^2-x^2},\ \dfrac{1}{\sqrt{a^2-x^2}} \text{ は } x = a\sin\theta \text{ とおく}}$

ex

(1) $\displaystyle\int_0^3 \sqrt{9-x^2}\,dx$

$x = 3\sin\theta$ とおく

$\dfrac{dx}{d\theta} = 3\cos\theta$

$dx = 3\cos\theta\,d\theta$

x	$0 \to 3$
θ	$0 \to \frac{\pi}{2}$

$-\frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{2}$ で考える

$$\int_0^{\frac{\pi}{2}} \sqrt{9-9\sin^2\theta}\cdot 3\cos\theta\,d\theta$$

$$= 9\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

$$= 9\int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \frac{9}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

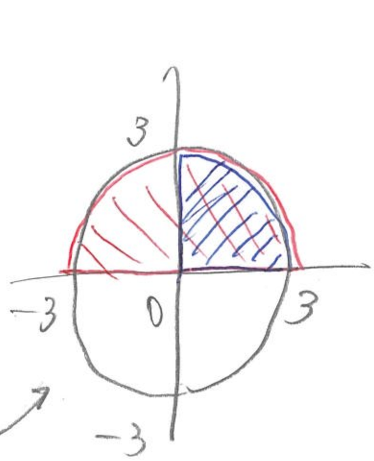
$$= \frac{9}{2}\cdot\frac{\pi}{2}$$

$$= \frac{9}{4}\pi$$

別解

$\boxed{y = \sqrt{9-x^2}}$

$x^2 + y^2 = 9$



$y = -\sqrt{9-x^2}$

$$\boxed{\int_0^3 \boxed{\sqrt{9-x^2}}\,dx}$$

$$= \frac{1}{4}\cdot 3^2\pi$$

$$= \frac{9}{3}\pi$$

$\sqrt{a^2-x^2}$, $\dfrac{1}{\sqrt{a^2-x^2}}$ の定積分 ②

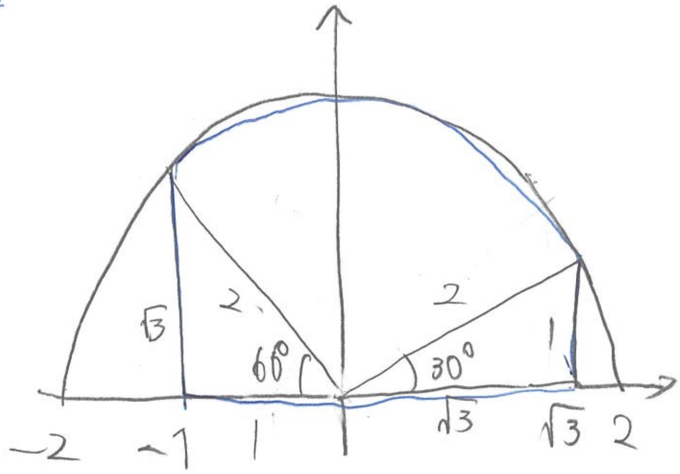
ex

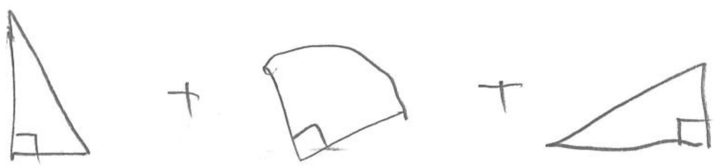
別解

(2) $\boxed{\displaystyle\int_{-1}^{\sqrt{3}} \sqrt{4-x^2}\, dx}$

$y = \sqrt{4-x^2}$

$x^2 + y^2 = 4$





$$= \frac{\sqrt{3}}{2} + \frac{1}{4}\cdot 2\cdot 2\cdot \pi + \frac{\sqrt{3}}{2}$$

$$= \underline{\pi + \sqrt{3}}$$

①	円 $= r^2\pi$
②	扇形 $= \dfrac{1}{2}\cdot r^2\theta$

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$\int \sqrt{a^2-x^2}\,dx$ は 円の面積の一部として求める.

$\dfrac{1}{x^2+a^2}$ の定積分 ①

$$\boxed{\frac{1}{x^2+a^2} \text{ は } x = a\tan\theta \text{ とおく}}$$

ex.

(1) $\displaystyle\int_{-1}^{1} \frac{dx}{3+x^2}$

$x = \sqrt{3}\tan\theta$ とおく

$$\frac{dx}{d\theta} = \frac{\sqrt{3}}{\cos^2\theta}$$

$$dx = \frac{\sqrt{3}}{\cos^2\theta}\,d\theta$$

x	$-1 \longrightarrow 1$
θ	$-\frac{\pi}{6} \longrightarrow \frac{\pi}{6}$

$-\frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{2}$ で考える

$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{3+3\tan^2\theta}\,\frac{\sqrt{3}}{\cos^2\theta}\,d\theta$$

$$= \frac{1}{\sqrt{3}}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \boxed{\frac{1}{1+\tan^2\theta}}\,\frac{1}{\cos^2\theta}\,d\theta$$

$$\left(\boxed{\cos^2\theta} = \boxed{\frac{1}{1+\tan^2\theta}} \right)$$

$$= \frac{1}{\sqrt{3}}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} d\theta$$

$$= \frac{\sqrt{3}}{9}\pi$$

$\frac{1}{x^2+a^2}$ の定積分 ②

ex

(2) $\displaystyle\int_1^2 \frac{dx}{\boxed{x^2-2x+2}}$

$\displaystyle = \int_1^2 \frac{dx}{\boxed{(x-1)^2+1}}$

$x-1=\tan\theta$ とおく

$\displaystyle\frac{dx}{d\theta} = \frac{1}{\cos^2\theta}$

$\displaystyle dx = \frac{1}{\cos^2\theta}\,d\theta$

x	$1 \longrightarrow 2$
θ	$0 \longrightarrow \frac{\pi}{4}$

$\displaystyle\int_0^{\frac{\pi}{4}} \frac{1}{\tan^2\theta+1}\cdot\frac{1}{\cos^2\theta}\,d\theta$

$\displaystyle = \int_0^{\frac{\pi}{4}} d\theta$

$\displaystyle = \frac{\pi}{4}$ //

$\frac{1}{ax^2+bx+c}$ の積分

$D>0 \longrightarrow$ 部分分数分解

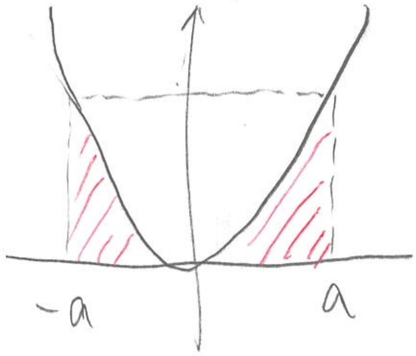
$D=0 \longrightarrow$ 積分

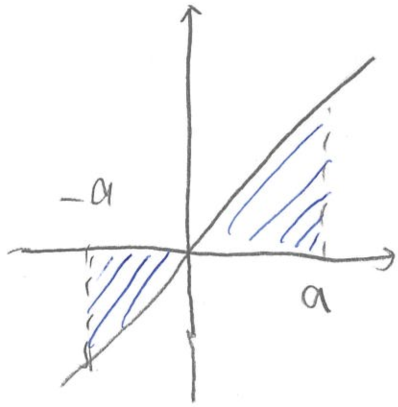
$D<0 \longrightarrow \tan\theta$

偶関数・奇関数

偶関数（y軸対称）のとき $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

奇関数（x軸 対称）のとき $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$





$f(-x) = f(x)$　　　　$f(-x) = -f(x)$

ex $\displaystyle\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos x\,(\sin x - x^3 + 1)\,dx$

偶　奇　奇

偶 × 偶 = 偶

偶 × 奇 = 奇

奇 × 奇 = 偶

$$= 2\int_{0}^{\frac{\pi}{3}} \cos x\,dx$$

$$= 2\left[\sin x\right]_{0}^{\frac{\pi}{3}}$$

$$= \sqrt{3}$$

$\boxed{18}$

部分積分法

ex 別解

$$\int fg\,dx = \overset{\text{そのまま}}{f}\,g - f\overset{\text{微分}}{g} + f\overset{\text{微分}}{g}$$

積分　積分　積分

$\boxed{19}$

(1)

$$\int_0^{\pi} x\sin x\,dx = \Big[x(-\cos x) - (-\sin x)\Big]_0^{\pi}$$

$$= \pi$$

(2)

$$\int_0^1 xe^x\,dx = \Big[xe^x - e^x\Big]_0^1$$

$$= 1$$

多項式 × 対数関数 には使えない

(3)

$$\cancel{\int_1^2 x\log x\,dx = \Big[\frac{1}{2}x^2\log x - \frac{1}{6}x^2\Big]_1^2}$$

$$\cancel{= 2\log 2 - \frac{2}{3} + \frac{1}{6}}$$

$$\cancel{= 2\log 2 - \frac{1}{2}}$$

$\boxed{19} \sim \boxed{22}$

定積分と微分 ①

$$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$$

$$f(x) \underset{\text{微分}}{\overset{\text{積分}}{\rightleftarrows}} F(x)$$

$$\int_a^x f(t)\,dt = F(x) - F(a)$$

両辺微分

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

定積分と微分 ②

$$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$$

$$\int_{定数}^{x} (tの式)\,dt \xrightarrow[微分]{} xの式$$

ex.

(1) $\int_2^x (3t^2-2t+4)\,dt \xrightarrow[微分]{} 3x^2-2x+4$

(2) $\int_{x}^{1} (3t^2-2t+4)\,dt \xrightarrow[微分]{} -(3x^2-2x+4)$

(3) $\int_0^x xe^t\,dt$

$= x\int_0^x e^t\,dt$

$\int_0^x xe^t\,dt$ tで積分する

であるから、
xを定数として変形

↓ xに関して
積の微分

$= (x)'\int_0^x e^t\,dt + x\left(\int_0^x e^t\,dt\right)'$

$= \int_0^x e^t\,dt + xe^x$

$= e^x - 1 + xe^x$ //

23 24

関数の決定

$f(x) = x + \int_0^{\pi} f(t)\sin t\,dt$ を満たす関数 $f(x)$ を求めよ.

(解答)

$\int_{数}^{数} f(t)\,dt = 数$ なので a とおける

$a = \int_0^{\pi} f(t)\sin t\,dt$ とおく.

$f(x) = x + a$

$f(t) = t + a$

$a = \int_0^{\pi} (\overset{f}{t + a}) \overset{g}{\sin t}\,dt$

$= \left[(t+a)(-\cos t) + (-\sin t)\right]_0^{\pi}$

$= \pi + a - (-a)$

$= \pi + 2a$

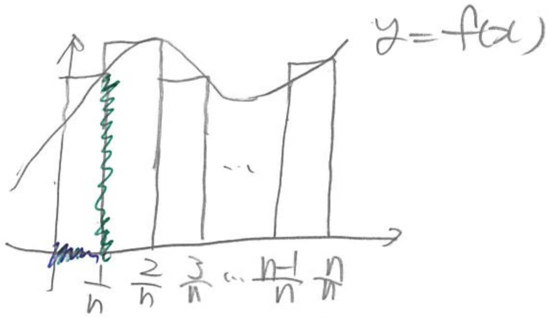
$a = -\pi$ $\quad\therefore\ f(x) = x - \pi$

$\boxed{26}$

区分求積法 ①

区分求積法…面積を細かく分割して、和の極限として求める方法

$$\boxed{\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)=\int_0^1 f(x)\,dx}$$

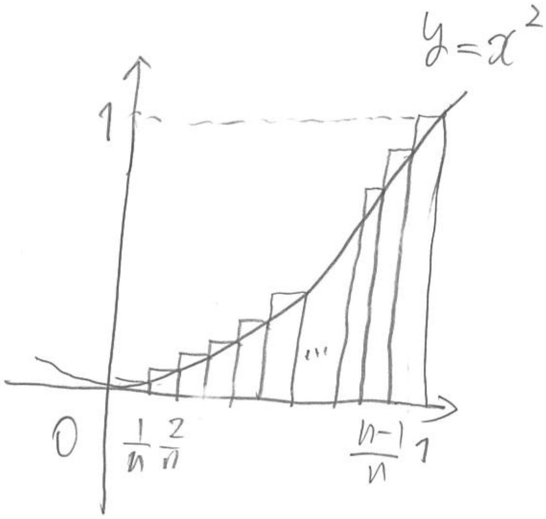


$$\int_0^1 f(x)\,dx=\lim_{n\to\infty}\left\{\underbrace{\frac{1}{n}}_{\text{ヨコ}}\underbrace{f\left(\frac{1}{n}\right)}_{\text{タテ}}+\frac{1}{n}f\left(\frac{2}{n}\right)+\cdots+\frac{1}{n}f\left(\frac{n}{n}\right)\right\}$$

$$=\lim_{n\to\infty}\frac{1}{n}\left\{f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\cdots+f\left(\frac{n}{n}\right)\right\}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)$$

ex.



$$S_n=\frac{1}{n}\left\{\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\cdots+\left(\frac{n}{n}\right)^2\right\}$$

$$=\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^2$$

$$=\frac{1}{n^3}\cdot\frac{1}{6}n(n+1)(2n+1)=\frac{1}{n^2}\frac{1}{6}(n+1)(2n+1)$$

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)=\underline{\frac{1}{3}}$$

$\boxed{27}$

区分求積法②

$$\boxed{\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)=\int_0^1 f(x)\,dx}$$

ex.

$$\lim_{n\to\infty}\left(\frac{1}{n^2}e^{\frac{1}{n}}+\frac{2}{n^2}e^{\frac{2}{n}}+\frac{3}{n^2}e^{\frac{3}{n}}+\cdots+\frac{n}{n^2}e^{\frac{n}{n}}\right)$$

$$=\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{n}e^{\frac{1}{n}}+\frac{2}{n}e^{\frac{2}{n}}+\cdots+\frac{n}{n}e^{\frac{n}{n}}\right)$$

$\frac{1}{n}$でくくる

$f(x)=xe^x$ とすると、

$f(x)$を出す

$$=\lim_{n\to\infty}\frac{1}{n}\left\{f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\cdots+f\left(\frac{n}{n}\right)\right\}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)$$

$$=\int_0^1 xe^x\,dx$$

$$=\left[xe^x-e^x\right]_0^1$$

$$=e-e-(-1)$$

$$=1$$

$$\boxed{\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}f\left(\frac{k}{n}\right)=\int_0^1 f(x)\,dx}$$

定積分と不等式

ex. nが2以上の自然数であるとき、次の不等式を証明せよ。

$$\log(n+1) < \underbrace{1+\frac{1}{2}+\frac{1}{3}+\cdots+\boxed{\frac{1}{n}}}_{②} < 1+\log n \quad ①$$

考え方

① $y=\boxed{\frac{1}{x}}$ のグラフを考える

② $k<x<k+1 \Longrightarrow \frac{1}{k+1}<x<\frac{1}{k}$ を利用する

③ $[k, k+1]$ で積分

(証明)

$k<x<k+1$ (k:自然数)

$$\frac{1}{k+1}<\frac{1}{x}<\frac{1}{k}$$

$$\frac{1}{k+1}\int_k^{k+1}dx < \int_k^{k+1}\frac{1}{x}dx < \frac{1}{k}\int_k^{k+1}dx$$

$$\underbrace{\frac{1}{k+1} < \int_k^{k+1}\frac{1}{x}dx}_{①} < \frac{1}{k} \quad ②$$

①

中央の形をつくるため

$$\sum_{k=1}^{n-1}\frac{1}{k+1} < \sum_{k=1}^{n-1}\int_k^{k+1}\frac{1}{x}dx$$

$$\boxed{\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}} < \int_1^n \frac{1}{x}dx$$

両辺に+1

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} < 1+\log n$$

②

中央の形をつくるため

$$\sum_{k=1}^{n}\int_k^{k+1}\frac{1}{x}dx < \sum_{k=1}^{n}\frac{1}{k}$$

$$\int_1^{n+1}\frac{1}{x}dx < \boxed{1+\frac{1}{2}+\cdots+\frac{1}{n}}$$

$$\left[\log x\right]_1^{n+1} < 1+\frac{1}{2}+\cdots+\frac{1}{n}$$

$$\log(n+1) < 1+\frac{1}{2}+\cdots+\frac{1}{n}$$

①②より

与式を満たす

(証明終)

$$\int_1^2+\int_2^3+\cdots+\int_{n-1}^n$$

$$=\int_1^n$$

29 30

積分法の応用

2曲線で囲まれた面積

$$S = \int_a^b (f - g)\,dx$$

共有点
と
上下

ex.

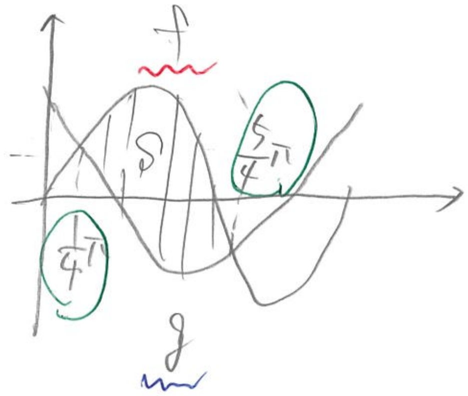
$y = \sin x$ (f)

$y = \cos x$ (g) $(0 \leqq x \leqq 2\pi)$ で囲まれた S

(解答)

$\sin x = \cos x$

$x = \frac{\pi}{4},\ \frac{5}{4}\pi$



$$S = \int_{\frac{\pi}{4}}^{\frac{5}{4}\pi} (\sin x - \cos x)\,dx$$

$$= \Big[-\cos x - \sin x\Big]_{\frac{5}{4}}^{\frac{5}{4}\pi}$$

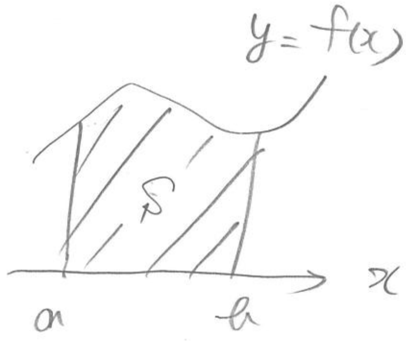
$$= \underline{2\sqrt{2}}$$

$\boxed{31} \sim \boxed{33}$

y軸方向への定積分①

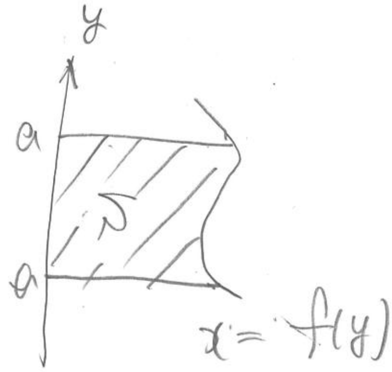
x軸 と $y=f(x)$

$$S=\int_a^b y\,dx=\int_a^b f(x)\,dx$$



y軸 と $x=f(y)$

$$S=\int_a^a x\,dy=\int_a^b f(y)\,dy$$



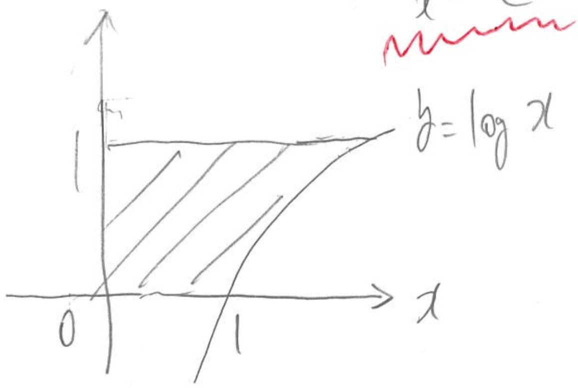
ex

$\begin{cases} y=\log x \\ x\text{軸} \\ y\text{軸} \\ y=1 \end{cases}$ で囲まれる S

> x軸方向で考えると
> 場合分けが必要。
> y軸方向で考えるとラク

(解答)

$x=e^y$



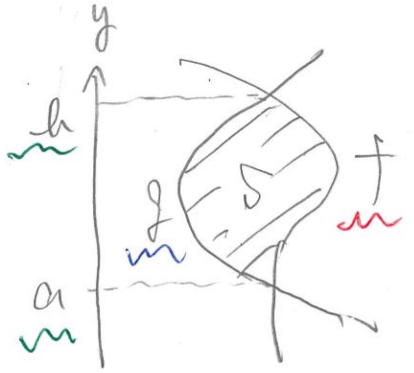
$$S=\int_0^1 x\,dy$$

$$=\int_0^1 e^y\,dy$$

$$=\underline{e-1}$$

y軸方向への定積分②

$$S=\int_a^b (f-g)\,dx$$



共有点

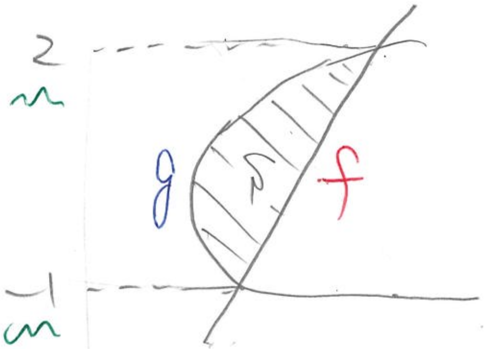
と

上下

ex.

$$\begin{cases} x=y^2 \\ x=y+2 \end{cases}$$ で囲まれる S

(解答)



$y^2=y+2$

$y^2-y-2=0$

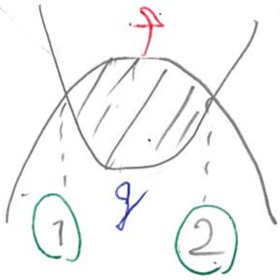
$(y-2)(y+1)=0$

$y=-1,\ 2$

f-g の y^2の係数

$$S=\int_{-1}^{2} -(y+1)(y-2)\,dy=\frac{3^3}{6}=\frac{27}{6}=\frac{9}{2}$$

復習



$y=x^2-4x+2$

$y=-x^2+2x-2$

$$\int_{①}^{②} -2(x-1)(x-2)\,dx=\frac{(2-1)^3}{6}$$

f-g の x^2の係数

$$=\frac{1}{6}$$

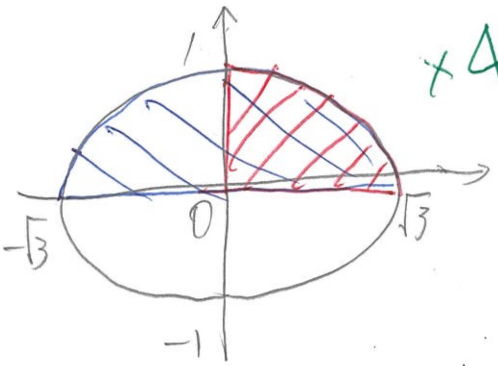
$\boxed{34}$

楕円① (陰関数)

ex. $\frac{x^2}{3}+y^2=1$ で囲まれた図形の面積S

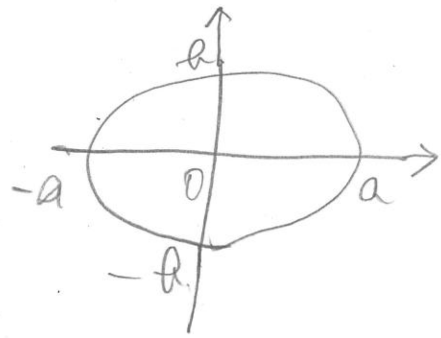
(解答)

$$\frac{x^2}{\sqrt{3}^2}+\frac{y^2}{1^2}=1$$



復習

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



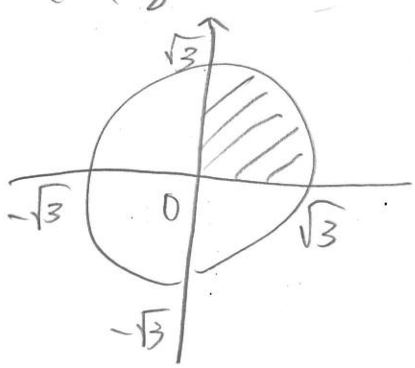
$$y^2=1-\frac{x^2}{3}$$

$$y=\sqrt{1-\frac{x^2}{3}}$$

$$\boxed{y=\frac{1}{\sqrt{3}}\sqrt{3-x^2}}$$

$$y=\sqrt{3-x^2}$$

$$x^2+y^2=3$$



$$S=4\cdot\boxed{\int_0^{\sqrt{3}}\frac{1}{\sqrt{3}}\sqrt{3-x^2}\,dx}$$

$$=\frac{4}{\sqrt{3}}\int_0^{\sqrt{3}}\sqrt{3-x^2}\,dx=\frac{4}{\sqrt{3}}\cdot\frac{1}{4}(\sqrt{3})^2\pi=\sqrt{3}\pi$$

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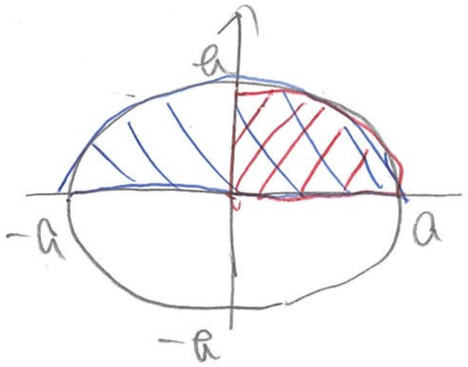
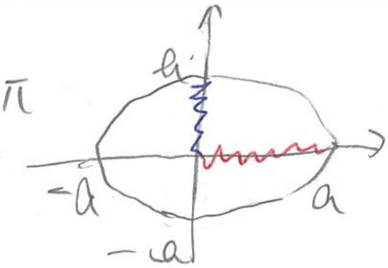
楕円③

ex　$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$　で囲まれるS　$(a>0,\ b>0)$

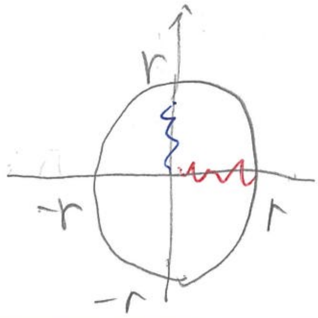
(解答)



楕円のS = $\underline{a}\,\underline{b}\,\pi$



円のS = $\underline{r}\,\underline{r}\,\pi$

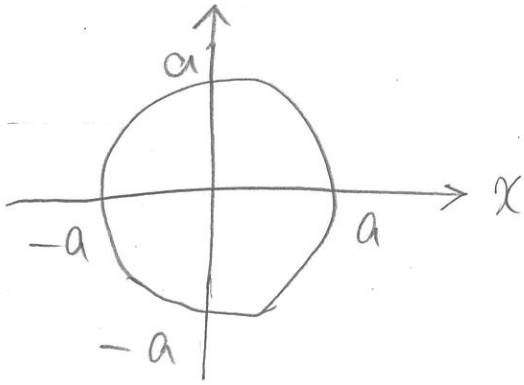


$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$y=\sqrt{a^2-x^2}$$

$$y^2=b^2\left(1-\frac{x^2}{a^2}\right)$$

$$x^2+y^2=a^2$$



$$y=b\sqrt{1-\frac{x^2}{a^2}}$$

$$\boxed{y=\frac{b}{a}\sqrt{a^2-x^2}}$$

$$S=\frac{1}{4}a^2\pi$$

$$S=4\times\boxed{\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx}$$

$$=\frac{4b}{a}\int_0^a\sqrt{a^2-x^2}\,dx=\frac{4b}{a}\cdot\frac{1}{4}a^2\pi=\underline{ab\pi}$$

媒介変数表示の曲線

$$\begin{cases} x = 3\cos\theta \\ y = 2\sin\theta \quad (0 \leqq \theta \leqq \pi) \end{cases}$$ で囲まれる S

(解答)

$$\int_{-3}^{3} y\,dx$$

$$\left(\frac{dx}{d\theta} = -3\sin\theta \qquad \begin{array}{c|c} \theta & 0 \longrightarrow \pi \\ \hline x & 3 \longrightarrow -3 \end{array} \right)$$

$$= \int_{-3}^{3} 2\sin\theta \cdot 3\sin\theta \cdot d\theta$$

$$= 6\int_{0}^{\pi} \sin^2\theta\, d\theta$$

$$= 3\int_{0}^{\pi} (1-\cos 2\theta)\, d\theta$$

$$= 3\left[\theta - \frac{1}{2}\sin 2\theta\right]_{0}^{\pi}$$

$$= \underline{3\pi}$$

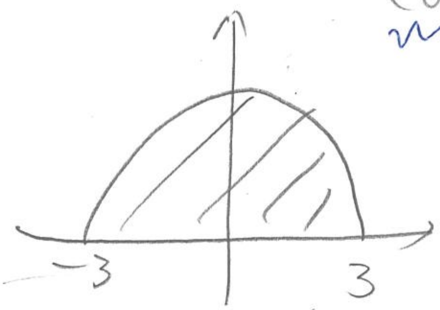
(別解)

$$\sin^2\theta + \cos^2\theta = 1$$

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

$$\frac{x^2}{9} + \frac{y^2}{4} = 1 \quad (-3 \leqq x \leqq 3)$$

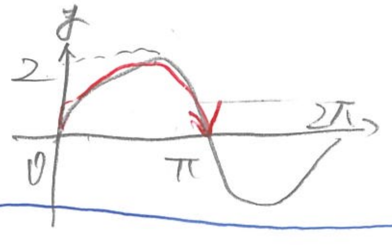
$$(0 \leqq y \leqq 2)$$



$$S = \frac{1}{2} \cdot 3 \cdot 2\pi = \underline{3\pi}$$

$$\begin{array}{c|c} \theta & 0 \longrightarrow \pi \\ \hline y & 0 \longrightarrow 0 \end{array}$$

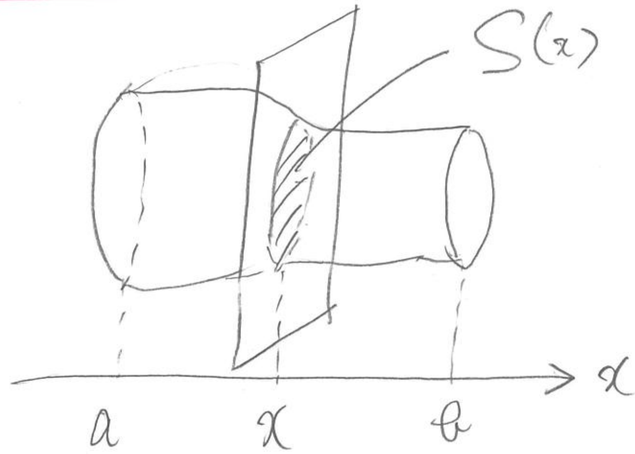
$$\begin{array}{c|c} \theta & 0 \longrightarrow \pi \\ \hline y & 0 \longrightarrow 2 \end{array}$$



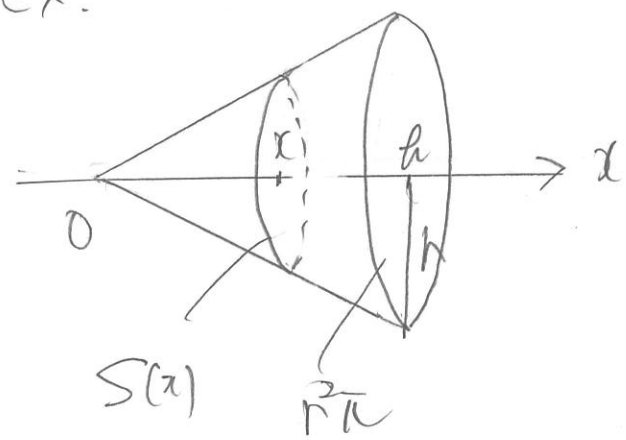
$\boxed{37}$

体積①

$$V = \int_a^b S(x)\,dx$$

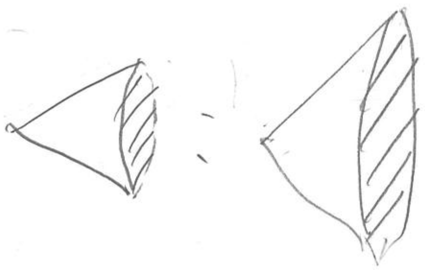


ex.



相似比　$m : n$

面積比　$m^2 : n^2$



$$S(x) : r^2\pi = x^2 : h^2$$

$$S(x) = \frac{x^2 r^2 \pi}{h^2}$$

$$V = \int_0^h S(x)\,dx$$

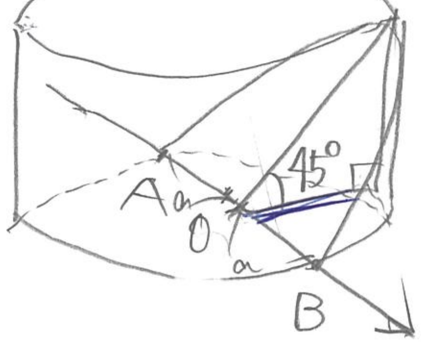
$$= \frac{r^2\pi}{h^2}\int_0^h x^2\,dx$$

$$= \frac{r^2\pi}{3h^2}\left[x^3\right]_0^h$$

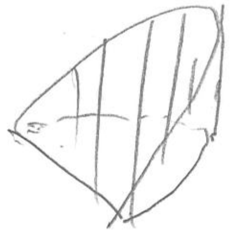
$$= \frac{1}{3}\cdot r^2\pi\cdot h$$

体積②

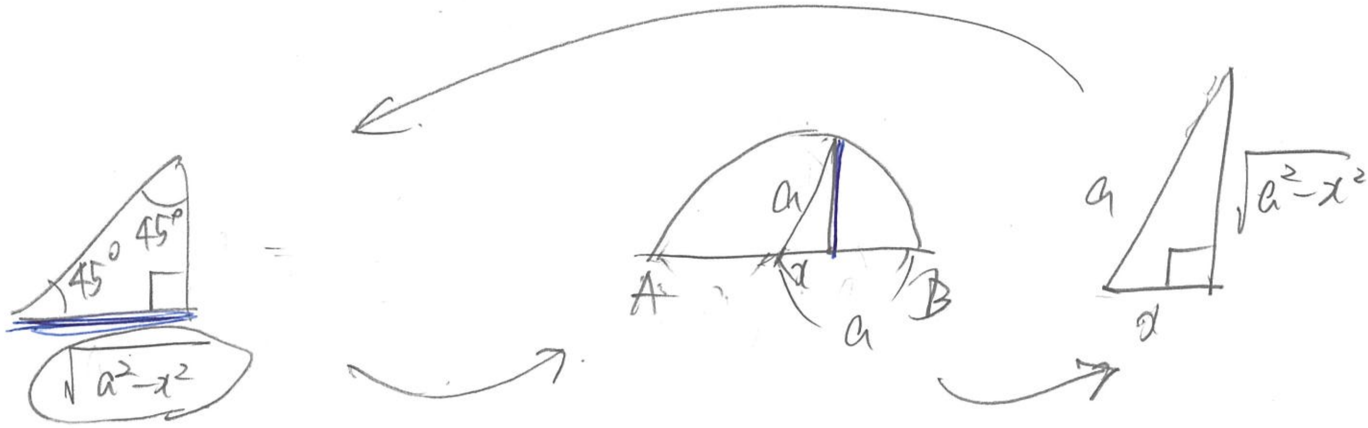
ex



底面の円の半径をaとする



Vを求めよ.



$$V=\int_{-a}^{a}\frac{1}{2}\left(\sqrt{a^2-x^2}\right)^2dx$$

$$=\frac{1}{2}\int_{-a}^{a}(a^2-x^2)dx$$

偶関数

$$=\left[a^2x-\frac{1}{3}x^3\right]_0^a$$

$$=a^3-\frac{1}{3}a^3$$

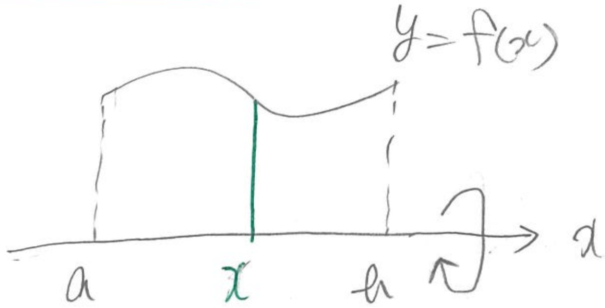
$$=\underline{\frac{2}{3}a^3}$$

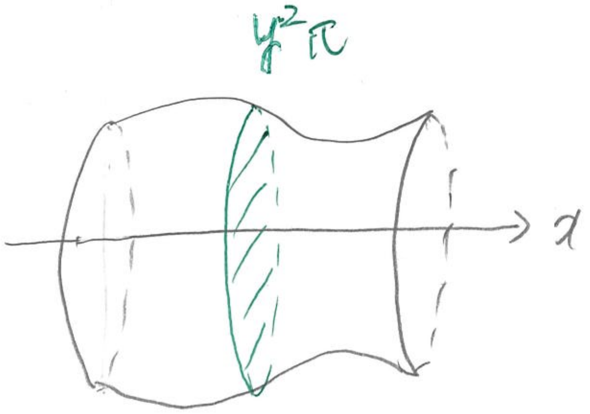
回転体の体積①

x軸の周りに1回転

$$V = \pi \int_a^b y^2 \, dx$$

$$= \pi \int_a^b f(x)^2 \, dx$$

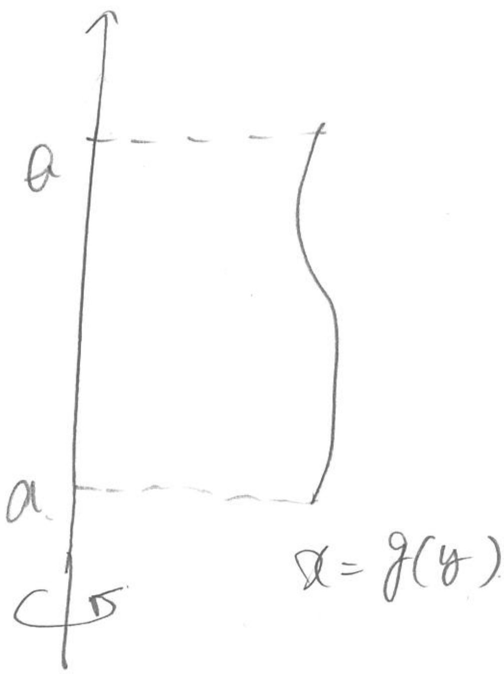




y軸の周りに1回転

$$V = \pi \int_a^b x^2 \, dy$$

$$= \pi \int_a^b g(y)^2 \, dy$$



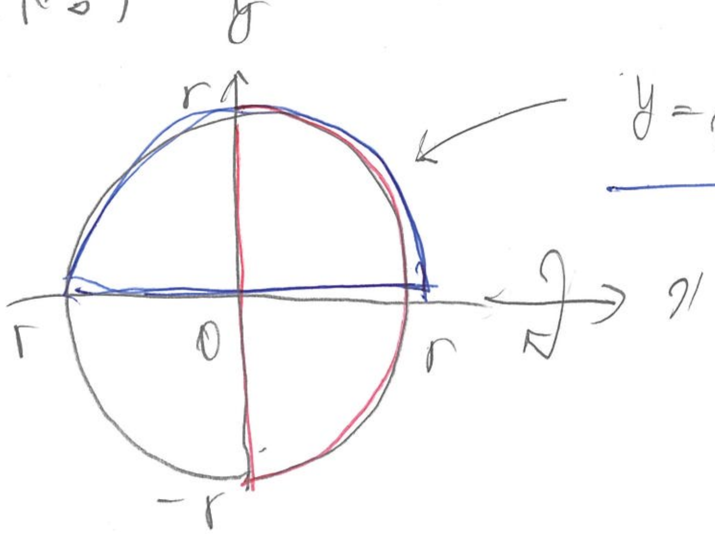
回転体の体積 ②

ex

円 $x^2+y^2=r^2$ で囲まれた図形が

x軸の周りに1回転してできる回転体の体積Vを求めよ.

(解答)



$y=\sqrt{r^2-x^2}$

$$V=2\times\boxed{\pi\int_0^r y^2\,dx}$$

$$=2\times\boxed{\pi\int_0^r (r^2-x^2)\,dx}$$

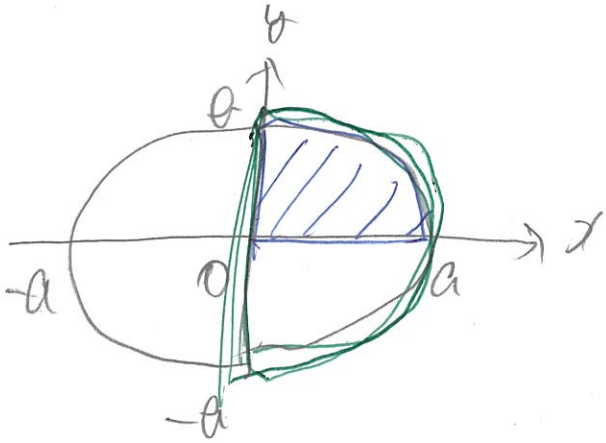
$$=2\pi\left[r^2x-\frac{1}{3}x^3\right]_0^r$$

$$=2\pi\left(r^3-\frac{1}{3}r^3\right)$$

$$=\frac{4}{3}\pi r^3$$

回転体の体積（楕円）

ex　$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



(1) 面積 S

$$S = 4\times \boxed{\frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx}$$

円で考える

$$= 4\times \frac{b}{a}\cdot\frac{1}{4}a^2\pi$$

$$= ab\pi$$

$$\frac{y^2}{b^2} = 1-\frac{x^2}{a^2}$$

$$y^2 = b^2\left(1-\frac{x^2}{a^2}\right)$$

$$= \frac{b^2}{a^2}(a^2-x^2)$$

$$y = \frac{b}{a}\sqrt{a^2-x^2}$$

(2) x軸の周りに1回転した V_x

$$V_x = 2\times \boxed{\frac{b^2}{a^2}\pi\int_0^a (a^2-x^2)\,dx}$$

$$= \frac{2\pi b^2}{a^2}\left[a^2x-\frac{1}{3}x^3\right]_0^a$$

$$= \frac{4}{3}\pi ab^2$$

(3) y軸の周りに1回転した V_y

$$V_y = \frac{4}{3}\pi a^2 b$$