三角関数①

【三角関数を含む方程式】

8 $0 \leqq \theta < 2\pi$ のとき，次の方程式を解け。

(1) $\sin\theta = \dfrac{\sqrt{3}}{2}$
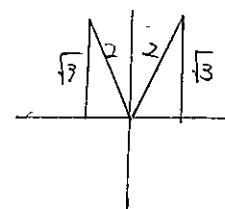
$\theta = \dfrac{\pi}{3}, \dfrac{2}{3}\pi$

(2) $2\cos\theta + 1 = 0$
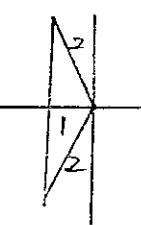
$\theta = \dfrac{2}{3}\pi, \dfrac{4}{3}\pi$

(3) $\cos\theta = -\dfrac{\sqrt{2}}{2}$
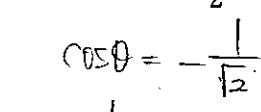
$\cos\theta = -\dfrac{1}{\sqrt{2}}$

$\theta = \dfrac{3}{4}\pi, \dfrac{5}{4}\pi$

(4) $\tan\theta = -\sqrt{3}$
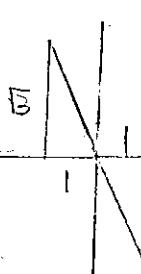
$\theta = \dfrac{2}{3}\pi, \dfrac{5}{3}\pi$

(5) $\sin\theta + 1 = 0$

$\sin\theta = -1$
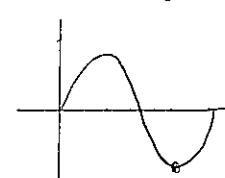
$\theta = \dfrac{3}{2}\pi$

(6) $\cos\theta = 0$
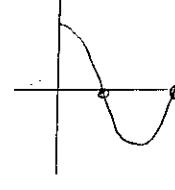
$\theta = \dfrac{\pi}{2}, \dfrac{3}{2}\pi$

【三角関数を含む不等式】

9 $0 \leqq \theta < 2\pi$ のとき，次の不等式を解け。

(1) $\cos\theta \leqq -\dfrac{1}{2}$
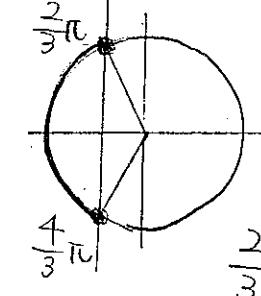
$\dfrac{2}{3}\pi \leqq \theta \leqq \dfrac{4}{3}\pi$

(2) $\sin\theta > \dfrac{1}{\sqrt{2}}$
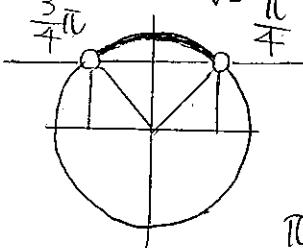
$\dfrac{\pi}{4} < \theta < \dfrac{3}{4}\pi$

(3) $\cos\theta > \dfrac{1}{2}$
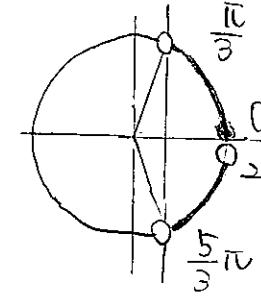
$0 \leqq \theta < \dfrac{\pi}{3}$

$\dfrac{5}{3}\pi < \theta < 2\pi$

(4) $\sin\theta \leqq \dfrac{\sqrt{3}}{2}$

$0 \leqq \theta \leqq \dfrac{\pi}{3}$

$\dfrac{2}{3}\pi \leqq \theta < 2\pi$

(5) $\tan\theta \leqq -1$

$\theta = \dfrac{3}{4}\pi, \dfrac{7}{4}\pi$

$\dfrac{1}{2}\pi < \theta \leqq \dfrac{3}{4}\pi$

$\dfrac{3}{2}\pi < \theta \leqq \dfrac{7}{4}\pi$

(6) $\tan\theta < \sqrt{3}$
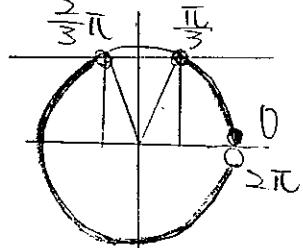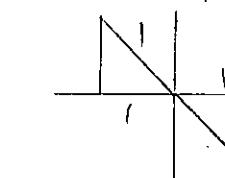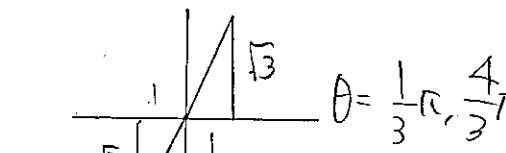
$\theta = \dfrac{1}{3}\pi, \dfrac{4}{3}\pi$
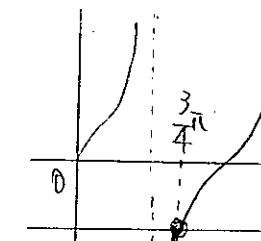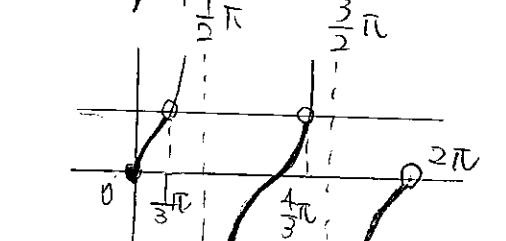
$0 \leqq \theta < \dfrac{\pi}{3}$

$\dfrac{\pi}{2} < \theta < \dfrac{4}{3}\pi$

$\dfrac{3}{2}\pi < \theta < 2\pi$

【三角関数を含む方程式[1]】

10 $0 \leqq \theta < 2\pi$ のとき，方程式 $\sin\left(\theta + \dfrac{\pi}{3}\right) = \dfrac{1}{\sqrt{2}}$ を解け。

$\theta + \dfrac{\pi}{3} = t$ とおく

$\sin t = \dfrac{1}{\sqrt{2}}$

$\begin{pmatrix} 0 \leqq \theta < 2\pi \\ \downarrow \\ \theta + \dfrac{\pi}{3} \leqq \theta + \dfrac{\pi}{3} < 2\pi + \dfrac{\pi}{3} \\ \downarrow \\ \dfrac{\pi}{3} \leqq t < \dfrac{7}{3}\pi \\ 60° \sim 420° \end{pmatrix}$

$t = \dfrac{\pi}{4}, \dfrac{3}{4}\pi$
$45° \quad 135°$
$\dfrac{9}{4}\pi$

$t = \dfrac{3}{4}\pi$ のとき， $t = \dfrac{9}{4}\pi$ のとき

$\theta + \dfrac{\pi}{3} = \dfrac{3}{4}\pi \qquad \theta + \dfrac{\pi}{3} = \dfrac{9}{4}\pi$

$\theta = \dfrac{5}{12}\pi \qquad \theta = \dfrac{23}{12}\pi$

【三角関数を含む方程式[2]】

11 $0 \leqq \theta < 2\pi$ のとき，方程式 $2\cos^2\theta + 3\cos\theta - 2 = 0$ を解け。

$\cos\theta = t$ とおく

$2t^2 + 3t - 2 = 0$

$(2t-1)(t+2) = 0$

$t = \cancel{-2}, \dfrac{1}{2}$
　　不適

$\begin{pmatrix} 0 \leqq \theta < 2\pi \\ \downarrow \\ -1 \leqq \cos\theta \leqq 1 \\ \downarrow \\ -1 \leqq t \leqq 1 \end{pmatrix}$

$t = \dfrac{1}{2}$ のとき

$\cos\theta = \dfrac{1}{2}$
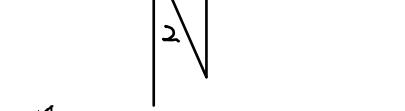
$\theta = \dfrac{\pi}{3}, \dfrac{5}{3}\pi$

【三角関数を含む関数の最大・最小】

12 $0 \leqq \theta < 2\pi$ のとき，関数 $y = -\sin^2\theta - \cos\theta + 1$ の最大値と最小値を求めよ。また、そのときの θ の値を求めよ。

$y = -(1 - \cos^2\theta) - \cos\theta + 1$

$y = \cos^2\theta - \cos\theta$

$\cos\theta = t$ とおく

$y = t^2 - t$

$= \left(t - \dfrac{1}{2}\right)^2 - \dfrac{1}{4}$

$\begin{pmatrix} 0 \leqq \theta < 2\pi \\ \downarrow \\ -1 \leqq \cos\theta < 1 \\ \downarrow \\ -1 \leqq t < 1 \end{pmatrix}$

よって

Max 2 $(t = -1)$

min $-\dfrac{1}{4}$ $\left(t = \dfrac{1}{2}\right)$

$y(-1) = 1 + 1$
$\quad = 2$

$\cos\theta = -1$
$\quad \theta = \pi$

$\cos\theta = \dfrac{1}{2}$
$\quad \theta = \dfrac{\pi}{3}, \dfrac{5}{3}\pi$

従って

Max 2 $(\theta = \pi)$

min $-\dfrac{1}{4}$ $\left(\theta = \dfrac{\pi}{3}, \dfrac{5}{3}\pi\right)$

【加法定理】

13 加法定理を用いて，次の三角関数の値を求めよ。

(1) $\sin 75° = \sin(45° + 30°)$

$= \sin 45° \cos 30° + \cos 45° \sin 30°$

$= \dfrac{1}{\sqrt{2}} \cdot \dfrac{\sqrt{3}}{2} + \dfrac{1}{\sqrt{2}} \cdot \dfrac{1}{2}$

$= \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

$= \dfrac{\sqrt{6}+\sqrt{2}}{4}$
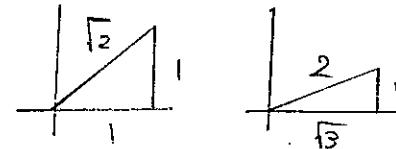
(2) $\cos 15° = \cos(45° - 30°)$

$= \cos 45° \cos 30° + \sin 45° \sin 30°$

$= \dfrac{1}{\sqrt{2}} \cdot \dfrac{\sqrt{3}}{2} + \dfrac{1}{\sqrt{2}} \cdot \dfrac{1}{2}$

$= \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

$= \dfrac{\sqrt{6}+\sqrt{2}}{4}$
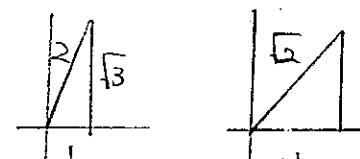
(3) $\tan 105° = \tan(60° + 45°)$

$= \dfrac{\tan 60° + \tan 45°}{1 - \tan 60° \tan 45°}$

$= \dfrac{\sqrt{3}+1}{1-\sqrt{3}}$

$= \dfrac{1+\sqrt{3}}{1-\sqrt{3}}$

$= \dfrac{(1+\sqrt{3})^2}{(1-\sqrt{3})(1+\sqrt{3})} = \dfrac{4+2\sqrt{3}}{-2}$
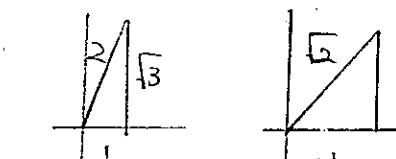
$= \dfrac{4+2\sqrt{3}}{1-3} = -2-\sqrt{3}$

【2直線のなす角】

14 2直線 $y=2x-1$，$y=\dfrac{1}{3}x+1$ のなす角 θ を求めよ。ただし，$0<\theta<\dfrac{\pi}{2}$ とする。

$\tan\alpha = 2$

$\tan\beta = \dfrac{1}{3}$

$\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \dfrac{\frac{5}{3}}{\frac{5}{3}}$

$= \dfrac{2-\frac{1}{3}}{1+\frac{2}{3}} = 1$

よって $\theta = \dfrac{\pi}{4}$
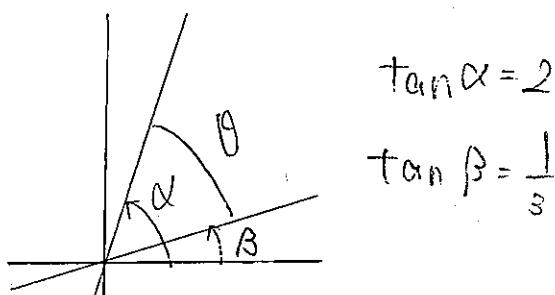
【加法定理の応用】

15 α の動径が第2象限，β の動径が第1象限にあり，$\sin\alpha = \dfrac{2}{3}$，$\cos\beta = \dfrac{3}{5}$ のとき，$\sin(\alpha-\beta)$ と $\cos(\alpha+\beta)$ を求めよ。

$x^2+4 = 9$ $\quad x = \sqrt{5}$

$x^2+9 = 25$ $\quad x = 4$

$\cos\alpha = -\dfrac{\sqrt{5}}{3}$
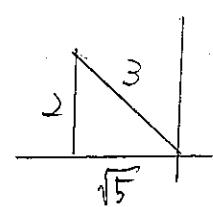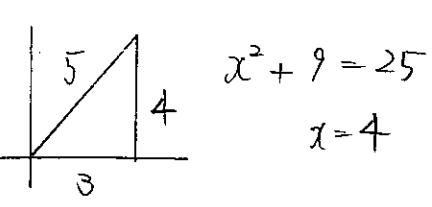
$\sin\beta = \dfrac{4}{5}$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= \dfrac{2}{3} \cdot \dfrac{3}{5} + \dfrac{\sqrt{5}}{3} \cdot \dfrac{4}{5}$

$= \dfrac{6+4\sqrt{5}}{15}$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= -\dfrac{\sqrt{5}}{3} \cdot \dfrac{3}{5} - \dfrac{2}{3} \cdot \dfrac{4}{5}$

$= \dfrac{-3\sqrt{5}-8}{15}$

【2倍角の公式・半角の公式】

16 $\dfrac{\pi}{2} < \alpha < \pi$ で $\sin\alpha = \dfrac{3}{5}$ のとき，次の値を求めよ。

(1) $\cos\alpha$
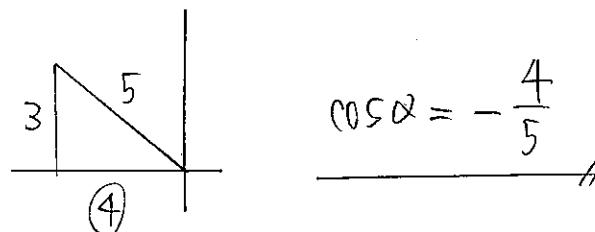
$\cos\alpha = -\dfrac{4}{5}$

(2) $\sin 2\alpha = 2\sin\alpha\cos\alpha$

$= 2 \cdot \dfrac{3}{5} \cdot \left(-\dfrac{4}{5}\right)$

$= -\dfrac{24}{25}$

(3) $\cos 2\alpha = 2\cos^2\alpha - 1$

$= 2 \cdot \dfrac{16}{25} - 1$

$= \dfrac{32}{25} - \dfrac{25}{25}$

$= \dfrac{7}{25}$

(4) $\sin\dfrac{\alpha}{2}$

$\sin^2\dfrac{\alpha}{2} = \dfrac{1-\cos\alpha}{2}$

$= \dfrac{1+\frac{4}{5}}{2}$

$= \dfrac{\frac{9}{5}}{2} = \dfrac{9}{10}$

$\sin\dfrac{\alpha}{2} > 0$ より

$\sin\dfrac{\alpha}{2} = \dfrac{3}{\sqrt{10}}$

$= \dfrac{3\sqrt{10}}{10}$

【2倍角の公式・半角の公式[2]】

17 次の値を求めよ。

(1) $\tan\alpha=3$ のとき，$\tan 2\alpha$ の値

$$\tan(\alpha+\alpha)=\frac{\tan\alpha+\tan\alpha}{1-\tan\alpha\tan\alpha}$$

$$=\frac{2\tan\alpha}{1-\tan^2\alpha}=\frac{6}{1-9}=-\frac{3}{4}$$

(2) $\frac{\pi}{2}<\alpha<\pi$ で $\cos\alpha=-\frac{2}{3}$ のとき，$\tan\frac{\alpha}{2}$ の値

$$\tan^2\frac{\alpha}{2}=\frac{1-\cos\alpha}{1+\cos\alpha} \qquad \oplus$$

$\tan\frac{\alpha}{2}>0$ より

$\tan\frac{\alpha}{2}=\sqrt{5}$

$$=\frac{1+\frac{2}{3}}{1-\frac{2}{3}}=\frac{\frac{5}{3}}{\frac{1}{3}}=5$$

【半角の公式】

18 半角の公式を使って，次の値を求めよ。

(1) $\sin\frac{\pi}{8}$

$$\sin^2\frac{\pi}{8}=\frac{1-\cos\frac{\pi}{4}}{2}$$

$$=\frac{1-\frac{1}{\sqrt{2}}}{2}$$

$$=\frac{\sqrt{2}-1}{2\sqrt{2}}=\frac{2-\sqrt{2}}{4}$$

$\sin\frac{\pi}{8}>0$ より

$\sin\frac{\pi}{8}=\frac{\sqrt{2-\sqrt{2}}}{2}$
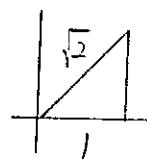
(2) $\cos\frac{3}{8}\pi$

$$\cos^2\frac{3}{8}\pi=\frac{1+\cos\frac{3}{4}\pi}{2}$$
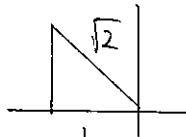
$$=\frac{1-\frac{1}{\sqrt{2}}}{2}$$

$\cos\frac{3}{8}\pi>0$ より

$$=\frac{\sqrt{2}-1}{2\sqrt{2}}=\frac{2-\sqrt{2}}{4}$$

$\cos\frac{3}{8}\pi=\frac{\sqrt{2-\sqrt{2}}}{2}$

【2倍角の公式と方程式】

19 $0\leqq\theta<2\pi$ のとき，次の方程式を解け。

(1) $\cos 2\theta+\sin\theta=1$

$1-2\sin^2\theta+\sin\theta=1$

$2\sin^2\theta-\sin\theta=0$

$\sin\theta(2\sin\theta-1)=0$

$\sin\theta=0 \qquad 2\sin\theta-1$

$\sin\theta=\frac{1}{2}$

$\theta=0,\pi$

$\theta=\frac{\pi}{6},\frac{5}{6}\pi$

よって
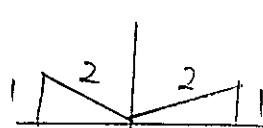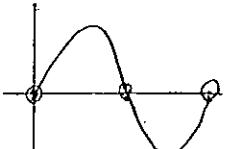
$\theta=0,\pi,\frac{\pi}{6},\frac{5}{6}\pi$

(2) $\sin 2\theta+\cos\theta=0$

$2\sin\theta\cos\theta+\cos\theta=0$

$\cos\theta(2\sin\theta+1)=0$

$\cos\theta=0 \qquad 2\sin\theta+1=0$
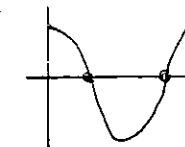
$\sin\theta=-\frac{1}{2}$

$\theta=\frac{1}{2}\pi,\frac{3}{2}\pi$
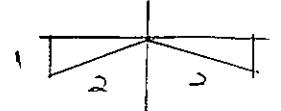
$\theta=\frac{7}{6}\pi,\frac{11}{6}\pi$

よって

$\theta=\frac{1}{2}\pi,\frac{3}{2}\pi,\frac{7}{6}\pi,\frac{11}{6}\pi$

【三角関数の合成】

20 次の式を $r\sin(\theta+\alpha)$ の形に表せ。ただし，$-\pi<\alpha<\pi$ とする。

(1) $\sqrt{3}\sin\theta+\cos\theta$

$=2\sin\left(\theta+\frac{\pi}{6}\right)$

(2) $\sin\theta-\cos\theta$

$=\sqrt{2}\sin\left(\theta-\frac{\pi}{4}\right)$

【三角関数の合成と最大・最小】

21 次の関数の最大値，最小値を求めよ。

$$y=-4\sin x+3\cos x$$

$=5\sin(x+\alpha)$
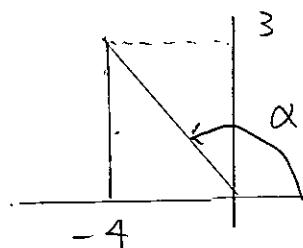
Max 5
min -5

【三角関数の合成と方程式】

22 $0\leqq x<2\pi$ のとき，次の方程式を解け。

$$\sin x+\sqrt{3}\cos x=1$$

$2\sin\left(x+\frac{\pi}{3}\right)=1$

$\sin\left(x+\frac{\pi}{3}\right)=\frac{1}{2}$

$x+\frac{\pi}{3}=t$ とおく

$\sin t=\frac{1}{2}$

$t=\frac{\pi}{6},\frac{5}{6}\pi$

$30° \quad 150°$

$\frac{13}{6}\pi$

$390°$

$$\left(\begin{array}{c}0\leqq x<2\pi \\ \downarrow \\ 0+\frac{\pi}{3}\leqq x+\frac{\pi}{3}<2\pi+\frac{\pi}{3} \\ \downarrow \\ \frac{\pi}{3}\leqq t<\frac{7}{3}\pi \\ 60°\sim420°\end{array}\right)$$

$t=\frac{13}{6}\pi$ のとき $\qquad t=\frac{5}{6}\pi$ のとき

$x+\frac{\pi}{3}=\frac{13}{6}\pi \qquad x+\frac{\pi}{3}=\frac{5}{6}\pi \qquad$ よって

$x=\frac{11}{6}\pi \qquad x=\frac{1}{2}\pi \qquad x=\frac{1}{2}\pi,\frac{11}{6}\pi$